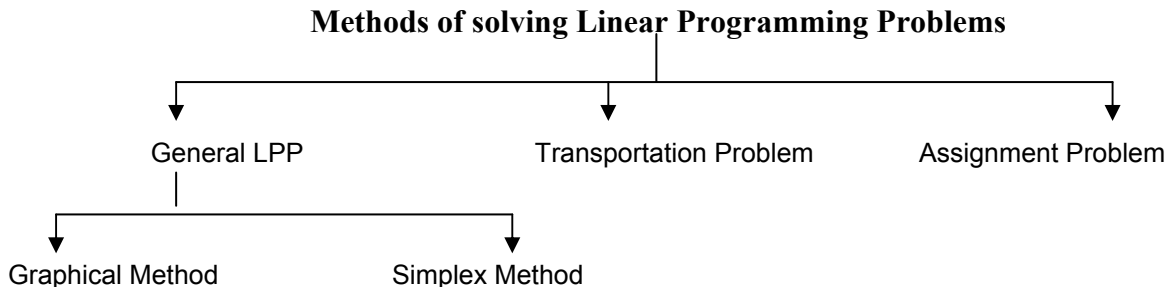


LINEAR PROGRAMMING

Linear Programming is a mathematical technique for determining the optimal allocation of resources achieving the specified objective when there are alternative uses of the resources like money, manpower, materials, machines and other facilities. The objective in resource allocation may be either cost minimization or profit maximization.



Steps in formulation of LPP:

- 1) The information stated in the problem is summarized in a table.
- 2) Identify the variables required & denote them by the symbols.
- 3) Formulate the objective function to be optimized (maximized or minimized) as a linear function of the variables.
- 4) Express the linear constraints mathematically in terms of variables.
- 5) Add the non-negative constraints from the consideration that –ve values of the variables do not have any valid significance.

1. A common mistake in LP is to use the gross profit per unit instead of the contribution margin per unit in a maximizing function for a manufacturing firm. Similarly, it is a mistake to use the full (absorption) cost of good sold per unit instead of the variable cost per unit in a cost minimizing function. These mistakes are usually caused by relying on a traditional income statement, which deducts cost of good sold from sales in determining gross profit.
2. Fixed costs or revenues that do not vary as units are produced or sold are not relevant in an LP problem.

Rules for Graphical Method:

- 1) Istly, formulate the LPP (steps are defined *supra*).
- 2) Graph each of the linear constraints by treating each linear inequation as an equation.
- 3) Identify the feasible region of the solution i.e. the area which satisfies all of the linear constraints simultaneously. Ensure that region is bounded. If the region is not bounded, either there are additional hidden conditions which can be used to bound the region or there is no solution to the problem.

Note: Study material of ICAI has suggested that the region drawn shall be bounded for both, problems of maximization as well as minimization.

- 4) Find the coordinates of the corner points of the feasible region as the optimum solution lies at one of the corners of the feasible region [Extreme Point Theorem].
- 5) Compute the value of the objective function at each point obtained in Step 4).
- 6) The set of values corresponding to the maximum (or minimum) value of the objective function is the solution of the linear programming problem when it is a maximization (or minimization) problem.

Assign some arbitrary value to objective function (this value can be obtained by taking the LCM of the co-efficients of x & y in the objective function) & draw the line for the equation representing the objective function. Common points obtained from drawing a line parallel to this line (touching feasible region along any edge) which is farthest from origin represents the optimal solution in problems of maximization.

Question 1: Maximize, $Z = 3x + 5y$
Subject to constraints,
 $x + 2y \leq 20$

$$\begin{aligned}x + y &\leq 15 \\y &\leq 8 \\x, y &\geq 0\end{aligned}$$

Solve the LPP under graphical method.

[Ans.: $Z^{\max} = 55$ when x is 10 and y is 5]

Question 2: Minimize, $Z = 2x + 3y$

Subject to constraints,

$$\begin{aligned}-x + 2y &\geq 4 \\x + y &\geq 6 \\x + 3y &\geq 9 \\x, y &\geq 0.\end{aligned}$$

Solve the LPP under graphical method.

[Ans.: $Z^{\min} = 10.2$ when x is $\frac{6}{5}$ and y is $\frac{13}{5}$]

Question 3: XYZ chemical company is producing two products A and B. The processing times are 3 hours and 4 hours per unit for A on operations one and two respectively and 4 hours and 2 hours per unit for B on operations on one and two respectively. The available time is 36 hours and 28 hours for operation one and two respectively. The product A can be sold at Rs. 3/- profit per unit and B at Rs. 8/- profit per unit. Solve for maximum profit programme by graphical & simplex method.

[Ans.: Profit is Rs.72, no. of units produced for A is nil & no. of units produced for B is 9]

Question 4 (Diet Problem): A diet for a sick person must contain at least 4000 units of vitamins, 50 units of minerals and 1400 units of calories. Two foods A and B are available at a cost of Rs. 4/- and Rs. 3/- per unit respectively. If one unit of A contains 200 units of vitamins, 1 unit of mineral and 40 calories and one unit of food B contains 100 units of vitamins, 2 units of minerals and 40 calories. What combination of food be used to have least cost?

[Ans.: Least cost is Rs. 110 when units purchased for A is 5 & B is 30]

Question 5: A manufacturer can produce two different products, A and B during a given time period. Each of these products requires four different manufacturing operations: Grinding, Turning, Assembling & Testing. The manufacturing requirements in hours per unit of product are given below for A & B

	A	B
Grinding	1	2
Turning	3	1
Assembling	6	3
Testing	5	4

The available capacities of these operations in hours for the given time period are: Grinding, 30; Turning, 60; Assembly, 200; Testing 200. The contribution to profit is Rs. 2 for each unit of A and Rs.3 for each unit of B. The firm can sell all that it produces at the prevailing market price. Formulate the problem as a linear programming model to maximize profit by graphical method.

[Ans.: Profit is Rs. 54 when no. of units produced for A is 18 & B is 6]

Question 6: A company produces two types of belts; X and Y. Profits on these types are Rs. 2 & Rs. 1.5 each belt respectively. **A belt of type X requires twice as much time as a belt of type Y. The company can produce at the most 1000 belts of type Y per day.** Material for 800 belts only per day is available. At the most 400 buckles for belts of type X & 700 for those of type Y are available per day. How many belts of each type should the company produce so as to maximize the profit?

[Hint: Let time taken to make one belt of type Y is t , therefore total time available in one day is $1000t$. Now we know, type x requires twice as much time as a belt of type y therefore total time for producing both these belts is $2tx + yt$. Therefore, **$2x + y \leq 1000$**]

[Ans.: 200 belts of type X and 600 belts of type Y]

Question 7: A manufacturer of patent medicines is preparing a production plan on medicines, A & B. There are sufficient raw materials available to make 20000 bottles of A and 40000 bottles of B, but there are only 45000 bottles into which either of the medicines can be put. **Further, it takes 3 hours to prepare enough material to fill 1000 bottles of A, it takes 1 hour to prepare enough material to fill 1000 bottles of B & there are 66 hours available for this operation.** The profit is Rs. 8 per bottle for A & Rs. 7 per bottle for B. How should the

manufacturer schedule his production in order to maximize his profit?

[Hint: Equation for bold line will be $3x/1000 + y/1000 \leq 66$]

[Ans.: 10500 bottles of A and 34500 bottles of B, max. profit Rs. 325500]

Question 8: A company that produces soft drinks has a contract that requires that a minimum of 80 units of the chemical A and 60 units of the chemical B into each bottle of the drink. The chemicals are available in a prepared mix from two different suppliers. Supplier X_1 has a mix of 4 units of A & 2 units of B that costs Rs. 10, and supplier X_2 has a mix of 1 unit of A and 1 unit of B that costs Rs. 4. How many mixes from company X_1 and company X_2 should the company purchase to honour contract requirement & yet minimize cost? Solve by both graphical as well as Simplex Method.

[Hint: Study material of ICAI has suggested finding hidden conditions for minimization problem (only for Graphical Methods). So in this question there will be two extra equations viz., $x_1 \leq 30$ & $x_2 \leq 80$ to bound the region]

[Ans.: Cost will be Rs. 260 when co. purchases 10 mixes from X_1 & 40 mixes from X_2]

Question 9: In a chemical industry two products A and B are made involving two operations. The production of B also results in a by-product C. the product A can be sold at a profit of Rs. 3 per unit and B at a profit of Rs. 8 per unit. The by-product C has a profit of Rs. 2 per unit. Forecasts show that up to 5 units of C can be sold. The company gets 3 units of C for each unit of B produced. The manufacturing times are 3 h per unit for A on each of the operation one & two and 4h and 5h per unit for B on operation one and two respectively. Because the product C results from producing B, no time is used in producing C. The available times are 18h and 21h of operation one and two respectively. The company desires to know that how much A and B should be produced keeping C in mind to make the higher profit. Formulate LP model for this problem. (10 Marks) May/01

[Ans.: Maximize $Z = 3x_1 + 8x_2 + 2x_3$

Subject to constraints

$$3x_1 + 4x_2 \leq 18$$

$$3x_1 + 5x_2 \leq 21$$

$$x_3 \leq 5, x_3 = 3x_2$$

$$x_1, x_2, x_3 \geq 0$$

[Hint: Since the ratios of the number of units produced between B and C is 1:3, therefore $x_2 = \frac{1}{3}x_3$]

Question 10 (Agriculture Application): An agriculture has a farm with 125 acres. He produces Radish, Peas and Potato. Whatever he raises is fully sold in the market. He gets Rs.5 for Radish per kg. Rs. 4 for Peas per kg. & Rs. 5 for potato per kg. The average yield is 1,500 kg. of Radish per acre; 1,800 kg. of Peas per acre and 1,200kg. of Potato per acre. To produce each 100 kg. of Radish and Peas and to produce each 80kg. of potato, a sum of Rs.12.50 has to be used for manure. Labour required for each acre to raise the crop is 6 man days for Radish and Potato each and 5 man days for Peas. A total of 500 man days of labour at the rate of Rs.40 per man day are available.

Formulate this as a LPP model to maximize the Agriculturist's total profit.

(10 Marks) May/1997

[Ans: $Z = 7072.5x_1 + 6775x_2 + 5572.5x_3$

Z constraint in detail, $7500x_1 + 7200x_2 + 6000x_3 - 187.5x_1 - 225x_2 - 187.5x_3 - 240x_1 - 200x_2 - 240x_3$

Subject to constraints,

$$x_1 + x_2 + x_3 \leq 125$$

$$6x_1 + 5x_2 + 6x_3 \leq 500$$

$$x_1, x_2, x_3 \geq 0$$

Question 11: The budgeted data relating to two products manufactured by a Co. for a month are as under:

	Product A	Product B
Selling price	300	200
Variable manufacturing cost	160	60
Sales commission	60	40

Each unit of product incurs costs in the company's two departments P and Q. The total capacity available for the month under review is budgeted to be 1,400 hours in department P and 2,000 hours in department Q. The capacity costs amount to Rs. 14,000 and Rs. 20,000 respectively per month for P and Q irrespective of the level of usage made of it. The number of hours required in each of these departments to complete one unit of output is as under:

	A	B
Department P	2	4
Department Q	5	4

The maximum output which the company can sell in the month is restricted to 400 units of either of the products. You are required to formulate the Linear Programming (LP) model and solve it graphically to determine the optimal product mix and profit. (8 Marks) May/2004

[Ans.: Optimal Profit Rs. 7000]

Question 12: A company manufactures two products A & B, involving three departments- Machining, Fabrication, & Assembly. The process time, profit/unit and total capacity of each department is given in the following table:

	Manufacturing (Hrs.)	Fabrication (Hrs.)	Assembly (Hrs.)	Profit (Rs.)
A	1	5	3	80
B	2	4	1	100
Capacity	720	1800	900	

Set-up LPP to maximize profit. What will be the product mix at maximum profit level? (9 Marks) May/2005

[Ans.: $x = 120$ units & $y = 300$ units and the maximum profit is Rs. 39600]

Question 13: Distinguish between a slack variable & an artificial slack variable in linear programming.

(3 Marks) May/2003

Slack variable:

In order to convert every constraint of the type 'less than equal to' in a LP problem into an equality constraint, so that solution of the problem can be arrived, we add a variable to each such constraint. The variable so added in each constraint is known as slack variable. *Slack Variables represent idle or unused resources.* The contribution per unit of a slack variable is always taken as zero in objective function. A slack variable is always non negative.

Constraint $3x + 2y \leq 90$ can be written as $3x + 2y + s_1 = 90$; here s_1 is a slack variable and is +ve.

Artificial variable:

In order to convert constraints of the type 'greater than equal to' equality for finding the solution of the L. P. problem, we first subtract a surplus variable and then add a variable. This variable is also added in the constraints of the type 'equal to' to start with the initial feasible solution. The variable added in the constraints as explained above is known as artificial variable. Artificial variable is a *fictitious variable* and cannot have any physical or economic meaning. It is intentionally introduced to form a initial solution for further iterations. It has an infinitely large cost coefficient. Artificial variables are always positive.

Constraints $3x + 4y \leq 50$ & $2x + 6y = 40$ can be written as $3x + 4y - s_1 + A_1 = 50$ & $2x + 6y + A_2 = 40$; Here s_1 is a surplus variable and A_1 & A_2 are artificial variables. s_1 , A_1 & A_2 are +ve.

Rules for Simplex Method:

- 1) Istly, formulate the LPP (steps are defined *supra*).
- 2) Add Slack Variable. (The $\leq$ type inequality can be transformed into equalities by addition of slack variables.)
- 3) Deduct Surplus Variable & Add Artificial Slack Variable. (This is done to convert $\geq$ type inequalities into equalities.)

[Similar to Slack Variable, Surplus Variable is subtracted to convert inequality into equality but as it has -1 as its coefficient it cannot be directly used as basic variable. To specify a basic variable an artificial variable is generally added with an infinitely large cost coefficient (M)]

For Initial Simplex tableau:

3.1 Setup the Simplex tableau.

Fixed Ratio	Profit/Cost Per unit (C_B)	Coefficient values from constraint equations						Quantity (b_i)	Replacement Ratio
		C_j	x_1	x_2	s_1	s_2	A_1		
	0				1		0		
	M				0		1		
	Z_j				0		M		
	Net Evaluation Row ($C_j - Z_j$)				0		0		

Where, s_1 is slack variable, s_2 is surplus variable & A_1 is Artificial Slack Variable.

- 3.2 In the first row i.e. C_j , we are supposed to write coefficients of variables in the objective function.
- 3.3 Basic Variables (a.k.a. Program Variables): If artificial slack variables (A_1, A_2, A_3 , etc.) are available it is artificial slack variables otherwise it is slack variables (s_1, s_2, s_3 , etc.).
- 3.4 Profit/Cost per unit (C_B): Coefficients of basic variables in objective function.
- 3.5 Write unit matrix as coefficients of basic variables.
- 3.6 In Z_j row, below basic variables, copy Profit/Cost per unit & write zeroes in Net Evaluation Row (NER).

Step 3.6 is basically part of optimality test but has been deliberately written earlier just to do questions easily.

- 3.7 Fill the table below non-basic variables by coefficients values from constraint equations.
- 3.8 Quantity column (b_i) (a.k.a. solution values): These represent constraint values written in constraint equations.
- 3.9 Optimality Test:
 - a) NER: Add the products of Profit/Cost per unit (C_B) with respective column coefficients & write it in Z_j row i.e., $Z_j = \sum C_B \cdot a_{ij}$, where a_{ij} are the matrix element in the i th row and j th column. Now subtract it from objective function's coefficient of respective column (C_j) i.e. $C_j - Z_j$
 - b) For maximization problems: Check that whether there is any +ve NER, *it indicates the magnitude of opportunity cost of not including 1 unit of respective column variables (in other words, it represents the net profit which would result from introducing one unit of variable to the product).*
For minimization problems: Check that whether there is any -ve NER.
[Note]: If all values of NER are other than positive (or other than negative) in case of maximization (or minimization) problem the calculated simplex table is optimal one & no further iterations are required, otherwise step 3.10 will be performed.]

Deriving a Revised Tableau for Improved/Optimal Solution:

- 3.10 Selection of entering variable: For maximization problem - The column with **highest +ve NER** is key column (In case of tie choose arbitrarily). For minimization problem - The column with **highest -ve NER** is key column (In case of tie choose arbitrarily). The variable heading this column is entering variable.
- 3.11 Selection of leaving variable: Divide Quantity (b_i) by corresponding elements of key column & write it in Replacement Ratio column. The row having **least non-negative** replacement ratio is key row, & the current variable for this row is leaving variable. The element at the intersection of key row & the key column is known as pivot/key element & is encircled.
- 3.12 Fixed ratio: This is calculated for all rows (variables) other than Key row.
 Fixed ratio = row element in key column $\div$ key no.
- 3.13 Setup a new updated simplex table.
 - a. Basic Variables:
 For key row – Leaving variable will be replaced by entering variable.
 For Non key rows - Variable will remain same as that of preceding simplex table.
 - b. Profit/Cost per unit (C_B): Coefficients of basic variables in objective function.
 - c. Write unit matrix as coefficients of basic variables.
 - d. In Z_j row, below basic variables, copy Profit/Cost per unit & write zeroes in Net Evaluation Row (NER).
 - e. Key Row [for non-basic variable column & quantity column (b_i)]: Divide all the nos. in the key row by the key no.
 - f. Non Key Rows [for non-basic variable column & quantity column (b_i)]: Old row no. – (Corresponding no. in key row $\times$ Corresponding fixed ratio).
 - g. Perform Optimality test (Same as Step 3.9)

- Marginal Value of Resource (a.k.a. shadow price or opportunity cost) is the value of NER under slack variables (s_1, s_2, s_3 , etc.). From the optimal (or other) solution profit gets reduced (or cost gets increased) by the amount this value is multiplied units of no. of units of such slack variables introduced.
- Fractions of simplex table must be retained & should not be decimalized.
- In situations where inequalities are like $0 \leq X \leq 30$, lower bounds shall be introduced i.e. we shall introduced a new variable like $x_1 = X + 30$ so computations can be made easy.

Question 14: The final simplex table for the problem is given below:

$$\begin{aligned} \text{Maximize } Z &= 3x_1 + 4x_2 + x_3 \\ \text{Subject to } x_1 + 2x_2 + 3x_3 &\leq 90 \quad (\text{constraint for operation 1}) \\ 2x_1 + x_2 + x_3 &\leq 60 \quad (\text{constraint for operation 2}) \\ 3x_1 + x_2 + 2x_3 &\leq 80 \quad (\text{constraint for operation 3}) \end{aligned}$$

Programme	Profit	Quantity	3	4	1	0	0	0
			x_1	x_2	x_3	s_1	s_2	s_3
x_2	4	40	0	1	10/6	4/6	-1/3	0
x_1	3	10	1	0	-1/3	-1/3	2/3	0
s_3	0	10	0	0	8/6	8/6	-10/6	1
	(Cj - Zj)		0	0	-28/6	-10/6	-2/3	0

Find the solution, maximum profit, idle capacity and the loss of total contribution of every one unit reduced from the right hand side of the constraints.

Question 15: Three grades of coal A, B and C contains phosphorus and ash as impurities. In a particular industrial process, fuel up to 100 ton (maximum) is required which could contain ash not more than 3% and phosphorus not more than 0.03%. It is desired to maximize the profit while satisfying these conditions. There is an unlimited supply of each grade. The percentage of impurities and the profits of each grades are as follows:

Coal	Phosphorus (%)	Ash (%)	Profit in Rs. (Per ton)
A	0.02	3.0	12.00
B	0.04	2.0	15.00
C	0.03	5.0	14.00

You are required to formulate the Linear Programming (LP) model to solve it by using simplex method to determine optimal product mix and profit. (11 Marks) Nov/2005

[Ans.: The optimal solution is $X_1 = 40$, $X_2 = 40$ & $X_3 = 20$ with maximum $Z = 1360$]

Question 16: A gear manufacturing company makes two types of gears – A and B. Both gears are processed on 3 machines, Hobbing M/c, Shaping M/c and Grinding M/c. The time required by each gear and total time available per week on each M/c is as follows:

Machine	Gear (A) (Hours)	Gear (B) (Hours)	Available Hours
Hobbing M/c	3	3	36
Shaping M/c	5	2	60
Grinding M/c	2	6	60

Other data:

Selling price (Rs.) 820 960

Variable cost (Rs.) 780 900

Determine the optimum production plan and the maximum contribution for the next week by simplex method.

The initial table is given below:

			C_j	40	60	0	0	0
		Qty.						
C_j	Variable		X_1	X_2	X_3	X_4	X_5	
0	X_3	36	3	3	1	0	0	
0	X_4	60	5	2	0	1	0	
0	X_5	60	2	6	0	0	1	

[Ans.: Optimum $Z = 660$ with $X_1 = 40$ and $X_2 = 60$]

(7 Marks) May/2007

Question 17: Solve by Simplex Method:

Minimize, $Z = 20x_1 + 10x_2$
 Subject to, $x_1 + x_2 \geq 10$
 $3x_1 + 2x_2 \geq 24$
 $x_1, x_2 \geq 0$

(Nov. 1987)

[Ans.: $Z^{\min} = 120$ when x_1 is 0 and x_2 is 12]

Miscellaneous Questions

Question 18 (Transportation Problem): The following matrix gives the unit cost of transporting the product from production plant P_1, P_2 & P_3 to destinations W_1, W_2 & W_3 . Plants P_1, P_2 & P_3 have a maximum production of 65, 24 and 111 respectively and destinations D_1, D_2 & D_3 must receive at least 60, 65 & 75 units respectively:

From \ To	D_1	D_2	D_3	Supply
P_1	400	600	800	65
P_2	1,000	1,200	1,400	24
P_3	500	900	700	111
Demand	60	65	75	200

You are required to formulate the above as a linear programming problem.

(Only formulation is needed. Please do not solve).

(9 Marks) Nov/08-New Course

[Ans.: Maximize, $Z = 400x_{11} + 600x_{12} + 800x_{13} + 1000x_{21} + 1200x_{22} + 1400x_{23} + 500x_{31} + 900x_{32} + 700x_{33}$

Subject to,
 $x_{11} + x_{12} + x_{13} = 65$
 $x_{21} + x_{22} + x_{23} = 24$
 $x_{31} + x_{32} + x_{33} = 111$
 $x_{11} + x_{21} + x_{31} = 60$
 $x_{12} + x_{22} + x_{32} = 65$
 $x_{13} + x_{23} + x_{33} = 75$
 where, $x_{ij} \geq 0$ ($i, j = 1$ to 3)

Question 19 (Transportation Problem): Transport Ltd. provides tourist vehicles of 3 types- 20-seater vans, 8-seater big cars & 5-seater small cars. These seating capacities are excluding the drivers. The company has 4 vehicles of the 20-seater van type, 10 vehicles of the eight-seater big car types 20 vehicles of the 5-seater small car types. These vehicles have to be used to transport employees of their client company from their residences to their offices and back. All the residences are in the same housing colony. The offices are at two different places, one is the Head Office and the other is the Branch. Each vehicle plies only one round trip per day, if residence to office in the morning and office to residence in the evening. Each day, 180 officials need to be transported in Route I (from residence to Head Office & back) and 40 officials need to be transported in Route II (from Residence to Branch office & back). The cost per round trip for each type of vehicle along each route is given below.

	20-seater vans	8-seater big cars	5-seater small cars
Route I- Residence-Head Office & Back	600	400	300
Route II- Residence-Branch Office & Back	500	300	200

You are required to formulate the information as LPP with objective of minimizing the total cost of hiring vehicles for the client company, subject to the constraints mentioned above. (10 Marks) May/2008

[Hint: Minimize, $Z = 600x_1 + 400x_2 + 300x_3 + 500y_1 + 300y_2 + 200y_3$

Subject to,
 $x_1 + y_1 \leq 4$
 $x_2 + y_2 \leq 10$
 $x_3 + y_3 \leq 20$
 $20x_1 + 8x_2 + 5x_3 = 180$
 $20y_1 + 8y_2 + 5y_3 = 40$
 $x_1, x_2, x_3, y_1, y_2, y_3 \geq 0$

Question 20 (Trim Problem): The Fine Paper Company produces rolls of paper used in cash registers. Each roll of paper is 500 ft. in length and can be produced in widths of 1,2,3 and 5 inch. The company's production

process results in 500 feet rolls that are 12 inches in width. Thus company must cut its 12 inch roll to the desired width. It has six basic cutting alternatives as follows:

<u>Cutting Alternative</u>	<u>No. of Rolls</u>				<u>Waste (inches)</u>
	<u>1"</u>	<u>2"</u>	<u>3"</u>	<u>5"</u>	
1	6	3	0	0	0
2	0	3	2	0	0
3	1	1	1	1	1
4	0	0	2	1	1
5	0	4	1	0	1
6	4	2	1	0	1

The maximum demand requirements for the four rolls are as follows:

<u>Roll Width (inches)</u>	<u>Demand Requirements(Rolls)</u>
1	3000
2	2000
3	1500
5	1000

The company wishes to minimize the waste generated by its production meeting its demand requirements. Formulate LP model.

[Ans.: Minimize, $x_3 + x_4 + x_5 + x_6$

Subject to,

$$6x_1 + x_3 + 4x_6 \leq 3000$$

$$3x_1 + 3x_2 + x_3 + 4x_5 + 2x_6 \leq 2000$$

$$2x_2 + x_3 + 2x_4 + x_5 + x_6 \leq 1500$$

$$x_3 + x_4 \leq 1000$$

$$x_j \geq 0; \text{ where } j = 1 \text{ to } 6]$$

Question 21: The Delhi Florist Company is planning to make up floral arrangements for the upcoming festival. The company has available the following supply of flowers at the costs shown:

<u>Type</u>	<u>Number Available</u>	<u>Cost per flower</u>
Red Roses	800	Rs. 0.20
Gardenias	456	Rs. 0.25
Carnations	4000	Rs. 0.15
White Roses	920	Rs. 0.20
Yellow Roses	422	Rs. 0.22

These flowers can be used in any of the four popular arrangements whose makeup and selling prices are as follows:

<u>Arrangement</u>	<u>Requirement</u>	<u>Selling Price</u>
Economy	4 red roses 2 gardenias 8 carnations	Rs. 6
May time	8 white roses 5 gardenias 10 carnations	Rs. 8
Spring colour	4 yellow roses 9 red roses 10 carnations 9 white roses	Rs. 10
Deluxe rose	6 yellow roses 12 red roses 12 white roses 12 yellow roses	Rs. 12

Formulate a LPP which allows the florist company to determine how many units of each arrangement should be made up in order to maximize profits assuming all arrangements are sold. Nov./90

[Ans.: Maximize, $3.5x_1 + 2.77x_2 + 3.58x_3 + 4.56x_4$

Subject to,

$$4x_1 + 9x_3 + 12x_4 \leq 800$$

$$2x_1 + 5x_2 \leq 456$$

$$8x_1 + 10x_2 + 10x_3 \leq 4000$$

$$8x_2 + 9x_3 + 12x_4 \leq 920$$

$$4x_2 + 6x_3 + 12x_4 \leq 422$$

$$x_j \geq 0; \text{ where } j = 1 \text{ to } 4]$$

Question 22: A manufacturer produces three products Y_1, Y_2, Y_3 from three raw materials X_1, X_2 and X_3 . The cost of raw materials X_1, X_2 and X_3 is Rs. 30, Rs. 50 and Rs. 120 per kg respectively and they are available in a limited quantity viz. 20 kgs of X_1 , 15 kgs of X_2 and 10 kgs of X_3 . The selling price of Y_1, Y_2 and Y_3 is Rs. 90, Rs. 100 and Rs. 120 per kg respectively. In order to produce 1 kg of Y_1 $\frac{1}{2}$ kg of X_1 , $\frac{1}{4}$ kg of X_2 and $\frac{1}{4}$ kg of X_3 are required. Similarly to produce 1 kg of Y_2 , $\frac{3}{7}$ kg X_1 , $\frac{2}{7}$ kg of X_2 and $\frac{2}{7}$ kg of X_3 and to produce 1 kg of Y_3 , $\frac{2}{3}$ kg of X_2 and $\frac{1}{3}$ kg of X_3 will be required.

Formulate the linear programming problem to maximize the profit.

(10 Marks) Nov./2000

[Ans.: Maximize, $Z = 32.50y_1 + 38.57y_2 + 46.67y_3$

Subject to $7y_1 + 6y_2 \leq 280$

$$21y_1 + 24y_2 + 56y_3 \leq 1260$$

$$21y_1 + 24y_2 + 28y_3 \leq 840$$

$$y_1, y_2, y_3 \geq 0]$$

Question 23 (Planning-production & financing) Consider a company that must produce two products over a production period of three months of duration. The company can pay for materials and labour from two sources:

The firm faces three decisions:

- (1) How many units should it produce of Product 1?
- (2) How many units should it produce of Product 2?
- (3) How much money should it borrow to support the production of the two products?

In making these decisions, the firm wishes to maximize the profit contribution subject to the conditions stated below:

- (i) Since the company's products are enjoying a seller's market, it can sell as many units as it can produce. The company would therefore like to produce as many units as possible subject to production capacity and financial constraints. The capacity constraints, together with cost and price data, are given in Table -1.

Capacity, Price and Cost data					
Product	Selling Price (Per unit)	Cost of Production (Per unit)	Requirement Hours per unit in Department		
			A	B	C
1	14	10	0.5	0.3	0.2
2	11	8	0.3	0.4	0.1
Available hours per production period of three months			500	400	200

- (ii) The available company funds during the production period will be Rs.3 lakhs.
- (iii) A bank will give loans up to Rs.2 lakhs per production period at an interest rate of 20 percent per annum provided the company's acid (quick) test ratio is at least 1 to 1 while the loan is outstanding. Take simplified acid-test ratio given by

$$\frac{\text{Surplus cash on hand after production} + \text{Accounts receivable}}{\text{Bank Borrowing} + \text{Interest accrued thereon}}$$

- (iv) Also make sure that the needed funds are made available for meeting the production costs.

Formulate the above as a Linear Programming Problem.

[Ans.: Maximize, $Z = 4x_1 + 3x_2 - 0.05x_3$

Subject to, $0.5x_1 + 0.3x_2 \leq 500$

$$0.3x_1 + 0.4x_2 \leq 400$$

$$0.2x_1 + 0.1x_2 \leq 200$$

$$10x_1 + 8x_2 - x_3 \leq 300000$$

$$-4x_1 - 3x_2 + 0.05x_3 \leq 300000$$

$$x_3 \leq 200000$$

$$x_3, x_3, x_3 \geq 0]$$

Question 24 (Blending Problems): A refinery makes 3 grades of petrol (A, B, C) from 3 crude oils (d, e, f)

Crude can be used in any grade but the others satisfy the following specifications.

Grade	Specifications	Selling Price per liter
A	Not less than 50% crude d Not more than 25% crude e	8.0
B	Not less than 25% crude d Not more than 50% crude e	6.5
C	No specifications	5.5

There are capacity limitations on the amount of the three crude elements that can be used;

Crude	Capacity	Price per litre
D	500	9.5
E	500	5.5
F	300	6.5

It is required to produce the maximum profit.

[Hint:

	D	E	F	Sell Price per liter
A	X_{A1}	X_{A2}	X_{A3}	8
B	X_{B1}	X_{B2}	X_{B3}	6.5
C	X_{C1}	X_{C2}	X_{C3}	5.5
Cost price per liter	9.5	5.5	6.5	
Capacity	500	500	300	

$$Z = 8(X_{A1} + X_{A2} + X_{A3}) - (9.5X_{A1} + 5.5X_{A2} + 6.5X_{A3}) + 6.5(X_{B1} + X_{B2} + X_{B3}) - (9.5X_{B1} + 5.5X_{B2} + 6.5X_{B3}) + 5.5(X_{C1} + X_{C2} + X_{C3}) - (9.5X_{C1} + 5.5X_{C2} + 6.5X_{C3})$$

$$\text{i.e. } Z = -1.5 X_{A1} + 2.5 X_{A2} + 1.5 X_{A3} - 3.0 X_{B1} + 1.0 X_{B2} - 4.0 X_{C1} - 1.0 X_{C3}$$

Subject to constraints,

$$\begin{aligned} X_{A1} &\geq 0.50 (X_{A1} + X_{A2} + X_{A3}) \\ X_{A2} &\leq 0.25 (X_{A1} + X_{A2} + X_{A3}) \\ X_{B1} &\geq 0.25 (X_{B1} + X_{B2} + X_{B3}) \\ X_{B2} &\leq 0.50 (X_{A1} + X_{A2} + X_{A3}) \\ X_{A1} + X_{B1} + X_{C1} &\leq 500 \\ X_{A2} + X_{B2} + X_{C2} &\leq 500 \\ X_{A3} + X_{B3} + X_{C3} &\leq 300 \\ X_{ij} &\geq 0 \text{ where } i = A, B \text{ \& } C \text{ \& } j = 1, 2 \text{ \& } 3 \end{aligned}$$

Question 25: A Computer Company produces three types of models, which are first required to be machined and then assembled. The time (in hours) required for these operations for each model is given below:

Model	Machine Time	Assembly Time
P III	20	5
P II	15	4
Celeron	12	3

The total available machine time and assembly time are 1,000 hours and 1,500 hours respectively. The data regarding the selling price and variable costs for the three models are:

	P III	PII	Celeron
Selling Price (Rs.)	3,000	5,000	15,000
Labour, material & other variable costs (Rs.)	2,000	4,000	8,000

The company has taken a loan of Rs. 50,000 from a Nationalized bank which is required to be repaid on 1st April, 2001. In addition, the company has borrowed Rs. 1,00,000 from XYZ Cooperative Bank. However, this bank has given its consent to renew the loan.

The Balance Sheet of this Company as on 31.3.01 is as follows :

Liabilities	Rs.	Assets	Rs.
Equity Share capital	1,00,000	Land	80,000
Capital Reserve	20,000	Building	50,000
Profit & Loss a/c	30,000	Plant & Machinery	1,00,000
Long term loan	2,00,000	Furniture & Fixtures	20,000
Loan from XYZ Co-op Bank	1,00,000	Vehicles	40,000

Loan from Nationalized Bank	50,000	Cash	2,10,000
Total	5,00,000	Total	5,00,000

The company is required to pay a sum of Rs. 15,000 towards the salary. Interest on long-term loan is to be paid every month @ 18% per annum. Interest on loan from XYZ Cooperative & Nationalized Banks may be taken as Rs. 1,500 per month. The company has already promised to deliver three P III, Two P II, and five Celeron type of Computers to M/s ABC Ltd. next month. The level of operation in the company is subject to the availability of cash next month.

The Company manager is willing to know that how many units of each model must be manufactured next month, so as to maximize the profit.

Formulate a linear programming problem for the above. (10 Marks) May/2001

[Hint: Maximize $Z = 1000x_1 + 1000x_2 + 7000x_3 - (\text{Rs. } 15,000 + \text{Rs. } 3,000 + \text{Rs. } 1,500)$

s.t.c, $20x_1 + 15x_2 + 12x_3 \leq 1000$

$5x_1 + 4x_2 + 3x_3 \leq 1500$

$2000x_1 + 4000x_2 + 8000x_3 \leq \text{Rs. } 140500$

Where $x_1 \geq 3, x_2 \geq 2, x_3 \geq 5$]

Question 26: A firm produces three products A,B & C. Its uses two types of raw materials I & II of which 5000 and 7500 units respectively are available. The raw material requirements per unit of product are given below:

Raw Material	Requirement per unit of product		
	A	B	C
I	3	4	5
II	5	3	5

The labour time for each unit of product A is twice that of product B and three times that of product C.

The entire labour force of the firm can produce the equivalent of 3000 units. The (marks) minimum demand of the three products is 600, 650 and 500 units respectively. **Also, the ratios of the number of units must be equal to 2:3:4.** Assuming the profits per unit of A,B & C as Rs. 50, 50 and 80 respectively.

Formulate the problem as linear programming model in order to determine the number of units of each product which will maximize the profit. (10 Marks) Nov./97

[Hint: 1. In the absence of a statement to the contrary, it is assumed that availability of labour is sufficient to produce 3000 units **of Product A.**

Let time taken to produce one unit of Product A is t , therefore total time available is $3000t$. Now we know, the labour time for each unit of product A is twice that of product B and three times that of product C, therefore total time for producing all three products is $tx_1 + tx_2/2 + tx_3/3$. Therefore, $x_1 + x_2/2 + x_3/3 \leq 3000$

2. Since the ratios of the number of units produced must be equal to 2:3:4, therefore

$$\frac{1}{2}x_1 = \frac{1}{3}x_2 \text{ and } \frac{1}{3}x_2 = \frac{1}{4}x_3 \text{ or } 3x_1 = 2x_2 \text{ and } 4x_2 = 3x_3]$$

Question 27; Renco Foundries is in the process of drawing up a Capital Budget for the next three years. It has funds to the tune of Rs. 100000 which can be allocated across the projects A, B, C, D and E. The net cash flows associated with an investment of Re.1 in each project are provided in the following table:

	Cash Flow at Time			
	0	1	2	3
From inv. A	-Re. 1	+Re. 0.5	+Re. 1	Re. 0
From inv. B	Re. 0	-Re. 1	+Re. 0.5	+Re. 1
From inv. C	-Re. 1	+Rs. 1.2	Re. 0	Re. 0
From inv. D	-Re.1	Re. 0	Re. 0	+Rs. 1.9
From inv. E	Re. 0	Re. 0	-Re. 1	+Rs. 1.5

Note: Time 0 = present, Time 1 = 1 year from now, Time 2 = 2 years from now, Time 3 = 3 years from now.

For example, Re. 1 invested in investment B requires a Re. 1 cash outflow at time 1 and returns Re. 0.50 at time 2 and Re. 1 at time 3.

To ensure that the firm remains reasonably diversified, the firm will not commit an investment exceeding Rs. 75000 in any project. The firm cannot borrow funds; therefore the cash available for investment at any time is limited to cash on hand. The firm will earn interest at 8% per annum by parking the uninvested funds in money market instruments. Assume that the returns from investments can be immediately reinvested. For example, the positive cash flow received from project C at time 1 can immediately be reinvested in project B.

Required: Formulate an LP that will "Maximize cash on hand at time 3".

(20 Marks) Nov./95

[Ans.: Maximize $Z = b + 1.9d + 1.5e + 1.08S_2$

Subject to Constraints,

$$\begin{aligned} a + c + d + S_0 &= 100000 \\ 0.5a + 1.2c + 1.08 S_0 &= b + S_1 \\ a + 0.5b + 1.08 S_1 &= e + S_2 \\ a, b, c, d, e &\leq 75000 \end{aligned}$$

Where a, b, c, d, e , and $S_i (i = 0, 1, 2) \geq 0$

Question 28: Let us assume that you have inherited Rs.1,00,000 from your father-in-law that can be invested in a combination of only two stock portfolio, with the maximum investment allowed in either portfolio set at Rs.75,000. The first portfolio has an average rate of return of 10%, whereas the second has 20% In terms of risk factors associated with these portfolios. The first has a risk rating of 4 (on a scale from 0 to 10), and the second has 9. Since you wish to maximize your return, you will not accept an average rate of return below 12% or a risk factor above 6. Hence, you then face the important question. How much should you invest in each portfolio?

Formulate this as a linear Programming. Problem and solve it by Graphic Method. (10 Marks) May/1999

[Hint: For finding bounded region apply 'true false' technique]

[Ans.: Co. should invest Rs. 60000 in first portfolio and Rs. 40000 in second portfolio to achieve the maximum average rate of return of Rs. 14000]

Question 29: A manufacturer of three products tries to follow a policy of producing those which contribute most to fixed cost and profit. However, there is a policy of recognizing certain minimum sales requirements. Currently these are:

Product	Unit per week
X	20
Y	30
Z	60

There are three producing departments. The product time in hour per unit in each department and the total times available for each week in each department are:

Department/Product	Time required per product(in hrs.)			Total hours available
	X	Y	Z	
1	0.25	0.20	0.15	420
2	0.30	0.40	0.50	1048
3	0.25	0.30	0.25	529

The contribution per unit of product X,Y,Z is Rs. 10.50, Rs. 9.00 and Rs. 8.00 respectively. The company has scheduled 20 units of X, 30 units of Y and 60 units of Z for production in the following week.

You are required to state:

- Whether the present schedule is an optimum one from a profit point of view and if it is not what it should be;
- The recommendations that should be made to the firm about their production facilities (following the answer to (a) above)

[Hint.: Solve it: Maximize, $Z = 10.5x_1 + 9x_2 + 8x_3 + 960$

Subject to, $0.25x_1 + 0.20x_2 + 0.15x_3 \leq 400$

$0.30x_1 + 0.40x_2 + 0.40x_3 \leq 1000$

$0.25x_1 + 0.30x_2 + 0.25x_3 \leq 500$

$x_1, x_2, x_3 \geq 0$; where $X = x_1 + 20, Y = x_2 + 30, Z = x_3 + 60$]

Question 30: The costs and selling prices per unit of two products manufacturing by a company are as under:

Product	A (Rs.)	B (Rs.)
Selling Price	500	450
Variable costs		
Direct Materials @ 25 per kg	100	100
Direct Materials @ 20 per kg	80	40
Painting @ Rs. 30 per hour	30	60
Variable Overheads:	190	175
Fixed costs @ Rs. 17.50/D.L.Hr	70	35
Total costs	470	410
Profit	30	40

In any month the maximum availability of input is limited to the following:

Direct Materials 480 kgs

Direct Labour hours	400 hours
Painting hours	200 hours

Required:

- (i) Formulate a linear programme to determine the production plan which maximizes the profits by using graphical approach.
- (ii) State the optimal product mix and the monthly profit derived from your solution in (i) above.
- (iii) If the company can sell the painting time as Rs. 40 per hour as a separate service, show what the modification will be required in the formulation of the linear programming problem. You are required to re-formulate the problem but not to solve. (11 Marks) Nov./08-Old Course

[Ans.: (i) Maximize, $30A + 40B$

subject to, $4A + 4B \leq 480$

$4A + 2B \leq 400$

$A + 2B \leq 200$

where, $A, B \geq 0$

& Maximum profit is Rs. 4400 when production of A is 40 units & production of B is 80 units.

(iii) Maximize, $20A + 20B$ {Painting hours @ 40 per hour is taken in computation}

subject to, $4A + 4B \leq 480$

$4A + 2B \leq 400$

$A + 2B \leq 200$

where, $A, B \geq 0$

[Assumptions: 1. In the above question Direct Material was written twice in question paper. It is assumed that it is due to printing mistake & Direct material @ 20 per kg is Direct Labour @ 20 per hour.

2. Fixed costs per unit are given, but no. of units based on which this overhead rate is calculated is not provided, so it is assumed that fixed costs vary as units are produced or sold & are relevant for this LP problem.]

Question 31: Explain the limitations of linear programming. (4 Marks) May/2004 & (5 Marks) Nov/2000

Answer: Important limitations of linear programming problems are as follows:

1. Linear Constraints: In some situations it is not possible to express both the objective function and constraints in linear form. E.g. Setup time is often non-linear constraint and is independent of the quantity produced.
2. Lack of Synergy emphasis: Joint interactions between some activities leads in comparatively less resource usage than sum of these quantities resulting from each activity being performed individually. From Linear programming it is not possible to handle such situations.
3. Fractions: Linear Programming may yield fractional valued answers for the decision variables, whereas it may happen that only integer values of the variables are logical. Rounding-off the values obtained may not result into an optimal solution. E.g. No. of units produced.
4. Uncertain Conditions: It is applicable only in static situations & objective function and the constraints equations should change during the period of study. Furthermore, these coefficients may actually be random variables, each with an underlying probability distribution for the values. Such problems can't be solved using LPP.
5. Heavy calculations: For large problems having many constraints, the computational difficulties are enormous, even when assistance of large digital computers is available.
6. Single Objective: LPP deals with problems that have a single objective. Real life problems may involve multiple and even conflicting objectives.

Question 32: Enumerate the industrial applications of linear programming. (4 Marks) May/2003

Answer: The industrial applications of linear programming are:

- Product mix problems
- Production scheduling
- Blending problems
- Transportation & distribution problems.

Question 33: What are practical applications of linear programming. (7 Marks) May/2007

Answer: Linear programming can be used to find optional solutions under constraints.

In production:

- product mix under capacity constraints to minimize costs/maximize profits along with marginal costing.
- Inventory management to minimize holding cost, warehousing / transporting from factories to warehouses etc.

Sensitivity Analysis: By providing a range of feasible solutions to decide on discounts on selling price, decisions to make or buy.

Blending: Optional blending of raw materials under supply constraints.

Finance: Portfolio management, interest/receivables management.

Advertisement mix: In advertising campaign – analogous to production management and product mix.

Assignment of personnel to jobs and resource allocation problems.

However, the validity will depend on the manager's ability to establish a proper linear relationship among variables considered.

Least Important Topics

<u>Situation</u>	<u>Treatment</u>
Multiple Optimal Situation	Graphical: If for a given question two or more extreme points yields same value of objective function it is multiple optimal solution. Simplex: For any optimal solution if NER of a non-basic variable is ZERO, it indicates that this variable can become incoming variable & there are more than 1 optimal solution.
Infeasibility	Graphical: If a solution satisfies all constraints and the non-negativity conditions it is called feasible solution, otherwise it is infeasible solution Simplex: If artificial variable persists in the optimal solution table as basic variable with non-zero quantity it is a infeasible solution.
Unboundedness	Graphical: For maximization type of LPP, unboundedness occurs when there is no constraint on the solution so that one or more of the decision variables can be increased indefinitely without violating any of the constraints. Simplex: If all Replacement Ratios are –ve i.e. Outgoing variable cannot be identified, it is an unbounded solution.
Degeneracy	Only Simplex: It occurs if there is a tie in the replacement ratios for determining outgoing variable; the next tableau would give a degenerate solution. In case of degeneracy, the outgoing variable should be selected on an arbitrary basis. The variable not selected as outgoing variable will bear "0" in the quantity column in the subsequent table.

Dual

Any LPP can be re-formulated into what is known as its dual. We can convert the Minimization Problem into Maximization Problem. This is known as duality.

Method of Deriving Dual:

- 1) Convert equations in inequalities (e.g. $x+y=10$ can be written as $x+y \geq 10$ & $x+y \leq 10$)
- 2) Convert LPP in Standard form i.e.
 - i) (Applicable for expert level only) All variable should be non-negative.
 - ii) For Maximization Type Problems → All constraints should be of " $\leq$ " type.
For Minimization Type Problems → All constraints should be of " $\geq$ " type.

E.g. Primal is:

Maximize, $Z = 20x + 30y$
Subject to constraints,

$$\begin{aligned} x + 2y &\leq 30 \\ 5x + 3y &\leq 60 \\ &\& x, y \geq 0 \end{aligned}$$

Dual will be:

Minimize, $Z = 30a + 60b$
Subject to constraints,

$$\begin{aligned} a + 5b &\geq 20 \\ 2a + 3b &\geq 30 \\ &\& a, b \geq 0 \end{aligned}$$

- iii) (Applicable for expert level only) If we have m constraints & n variables in primal (initial primal), we must have n constraints & m variables in Dual. Exception: It will not be so if we have equation in our initial LPP [i.e. we have processed Step i) & have converted equation in inequations]. In such case we will introduce new variable for similar variables [introduced by virtue of step i)].